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EXACT MULTIDIMENSIONAL SOLUTIONS OF THE NONLINEAR DIFFUSION EQUATION

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Exact nonnegative solutions are obtained for the nonlinear diffusion equation $u_t = \Delta(u^m)$, where Δ is the Laplacian in R^n, $n \geq 2$, and m is a positive constant. The solutions form an n-parameter family and correspond to initial data as a finite or an infinite measure. When $0 < m < 1$, its support is a hyperplane in R^n, while for $m > 1$ the initial measure is concentrated in a domain bounded by a second-order surface in R^l, $l < n$. The solutions generalize the known source-type solutions for the porous media equation and fast diffusion equation, but differ from them in that they are not self-similar. Examples of "nonsymmetric" exact solutions for the equation $u_t = \Delta \ln u$ with initial data of measure are presented. The properties of their symmetrization with time are discussed.

1. Let us consider the nonnegative solutions u(x, t) of the Cauchy problem

$$u_t = \Delta(u^m), \quad x = (x_1, \dots, x_n) \in R^n, \quad t > 0; \tag{1.1}$$

$$u(x, 0) = u_0(x), \quad x \in R^n, \tag{1.2}$$

where Δ is the Laplacian with respect to variables $x_1, \dots, x_n$; m = const > 0; and $u_0(x)$ is a specified nonnegative function. Equation (1.1), where t stands for time, and the space dimension is $n \leq 3$, arises in many applications. In particular, it describes the process of diffusion (thermal conduction) in a medium, whose diffusion coefficient (thermal conductivity) is a power function of concentration (temperature).

If $m > 1$, Eq. (1.1) is conventionally called the porous media equation. In this case the function u is identified with the density of a polytropic gas during its isentropic filtration through a uniform porous medium. The porous media equation is characterized by finiteness of the velocity with which disturbances propagate over the zero background [1], which markedly distinguishes it from the linear equation of heat conduction (m = 1). This occurs when the support of function u_0 does not coincide with the full space R^n. In this case problem (1.1), (1.2) has no classical solution. Extensive literature has been devoted to investigation of the generalized solutions of the problem (see, for example, [2] and the references there).

If $0 < m < 1$, Eq. (1.1) is called the fast diffusion equation. In particular, equations of this type appear in plasma physics and physics of semiconductors. The vanishing of nonnegative solution of the Cauchy problem over a finite time [3] is characteristic of the fast diffusion equation (see also [4] and references there).

2. We assume below that $n \geq 2$. Let us first consider problem (1.1), (1.2) for $m > 1$. Of special interest among its solutions are those of source type, which correspond to initial data like

$$u_0 = M\delta(x), \tag{2.1}$$

where M = const > 0, and $\delta(x)$ is the Dirac measure, and forms a one-parameter series with parameter M for fixed m and n. The problem (1.1), (1.2), (2.1) has been solved in [5]. The solution is self-similar and is expressed by elementary functions. This solution provides the generalized solution of the initial Cauchy problem (1.1) and (1.2) with the principal term of asymptotic expansion as $t \to \infty$, assuming that the norm in $L_1(R^n)$ of the finite function u_0 is finite and equals M [6]. In this case the law of mass conservation holds:

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$$\int\limits_{R^n} u(x,t)\,dx = \int\limits_{R^n} u_0(x)\,dx = M. \tag{2.2}$$

The uniqueness of the solution of the problem (1.1), (1.2), (2.1) has been established in [7]. The theorem of existence and uniqueness of the solution of problem (1.1), (1.2) with an arbitrary finite measure in the right-hand side of the condition (1.2) has been proved there. With the exception of the above, until recently no examples of exact solutions of the problem (1.1), (1.2) with initial data of measure have been known. An n-parameter series of the solutions is constructed below.

3. Further construction is based on a statement following from the results of [8, 9]. Let the vector function $X = (X_1, \ldots, X_n)$ of the variables $\xi = (\xi_1, \ldots, \xi_n)$, t form a sufficiently smooth solution of the system in the cylinder $Q_T = \{\xi, t : \xi \in \Omega, t \in (0.T)\}$

$$N^*\mathbf{X}_t = -m(m-1)^{-1}\nabla_\xi \left(|N|^{-m+1}\right). \tag{3.1}$$

Here $|N|$ is the Jacobian of the matrix N with the elements $N_{ij} = \partial X_i/\partial\xi_j (i, j = 1, \ldots, n)$. Let us assume that in the domain Ω of the space ξ the mapping $x = X(\xi, t)$ is one-to-one for each $t \in [0, T)$ and that $|N| > 0$ for $(\xi, t) \in Q_T$. Then the formulas

$$\mathbf{x} = \mathbf{X}(\xi, t), \qquad u = |N(\xi, t)|^{-1} \tag{3.2}$$

give parametric representation of the solution of Eq. (1.1).

We shall be concerned with the construction of an exact solution of system (3.1) of the form

$$X_i = \alpha_i(t) Y_i(\xi), \qquad i = 1, \ldots, n. \tag{3.3}$$

Substituting (3.3) into (3.1) yields:

$$\sum_{i=1}^{n} \alpha_i \frac{d\alpha_i}{dt} Y_i \frac{\partial Y_i}{\partial \xi_k} = -\frac{m}{m-1}\left(\prod_{i=1}^{n} \alpha_i\right)^{-m+1} \frac{\partial}{\partial \xi_k}(A^{-m+1}),$$

where $k = 1, \ldots, n$; $A = \det(\partial Y_i/\partial\xi_k)$. We make the unknown functions $\alpha_i(t)$ obey the system of equations

$$\alpha_1 \frac{d\alpha_1}{dt} = \ldots = \alpha_n \frac{d\alpha_n}{dt} = \frac{2m}{m-1}\left(\prod_{i=1}^{n} \alpha_i\right)^{-m+1}. \tag{3.4}$$

Then the relations connecting the functions $\alpha_i(t)$ and $Y_i(\xi)$ will be satisfied identically, if we set

$$A = \left(C - \sum_{i=1}^{n} Y_i^2\right)^{\frac{-1}{m-1}}$$

($C = \text{const} > 0$). The functions Y_i in this case may be chosen with a high degree of arbitrariness; however, their dependence on ξ in itself is of no particular interest, since to find the solution u(x, t) of Eq. (1.1) it suffices to know $|N|$ as a function of $x = X(\xi, t)$ and t. Using Eqs. (3.2) and (3.3), and the expression of A in terms of Y_i, we find:

$$u = \left(\prod_{i=1}^{n} \alpha_i\right)^{-1}\left(C - \sum_{i=1}^{n} \frac{x_i^2}{\alpha_i^2}\right)^{\frac{1}{m-1}}. \tag{3.5}$$

Here the functions $\alpha_i(t)$ are determined from the system (3.4). Below we will restrict our consideration to nonnegative solutions of the system, which guarantees nonnegativity of the function u(x, t).

Equation (3.5) prescribes the solution of Eq. (1.1) inside the ellipsoid $E_n(t)$ with center at the origin and semiaxes $C^{1/2}\alpha_i(t)$ $(i = 1, \ldots, n)$. Continuing the function u(x, t) by zero into the exterior of $E_n(t)$, we obtain a generalized solution of Eq. (1.1) defined over all space.

The system (3.4) possesses $n-1$ first integrals

$$\alpha_j^2 = \alpha_n^2 + \gamma_j, \qquad j = 1, \ldots, n-1, \tag{3.6}$$

where γ_j are constants. We will consider all values of γ_j to be nonnegative, which enables us to construct a solution of the system defined for any $t > 0$ and such that $a_n(0) = 0$. Moreover, without loss of generality one may assume that

$$\gamma_{n-1} \leq \gamma_{n-2} \leq \cdots \leq \gamma_1. \tag{3.7}$$

Let us determine the function $\alpha_n(t)$ for $t > 0$ as the inversion of the quadrature

$$\int_0^{\alpha_n} \beta^m \Big[\prod_{j=1}^{n-1} (\beta^2 + \gamma_j)\Big]^{\frac{m-1}{2}} d\beta = \frac{2mt}{m-1}, \tag{3.8}$$

and the values $a_j (j = 1, \ldots, n-1)$ as positive roots of Eqs. (3.6). Then the set of functions $\alpha_1(t), \ldots, \alpha_n(t)$ forms the solution of the Cauchy problem

$$\alpha_j(0) = \gamma_j^{1/2} \quad \text{for} \quad j = 1, \ldots, n-1; \qquad \alpha_n(0) = 0$$

for the system (3.4).

Let us assume that all inequalities (3.7) are strict, and $\gamma_{n-1} > 0$. In this case Eq. (3.5) determines an n-parameter family of solutions of Eq. (1.1). Here, according to (3.6), all semiaxes of the ellipsoid $E_n(t)$ are different, and $C^{1/2}\alpha_n(t)$ is the smallest of them. In view of (3.8), the asymptotic form of the function α_n in the limit $t \to 0$ is

$$\alpha_n = \Big[\frac{2m(m+1)}{m-1}\Big(\prod_{j=1}^{n-1} \gamma_j\Big)^{\frac{m-1}{2}} t\Big]^{\frac{1}{m+1}} + O(t^{\frac{2}{m+1}}).$$

Passing to the limit $t \to 0$ in the solution (3.5), we obtain

$$u(x,t) \to K_{m,0}\Big(\prod_{j=1}^{n-1} \gamma_j\Big)^{-1/2}\Big[\Big(C - \sum_{j=1}^{n-1} \frac{x_j^2}{\gamma_j}\Big)_+\Big]^{\frac{m+1}{2(m-1)}} \delta(x_n), \tag{3.9}$$

if $t \to 0$. Here $\delta(x_n)$ is the Dirac measure, r_+ denotes max(r, 0) and

$$K_{m,0} = 2\int_0^1 (1-\eta^2)^{\frac{1}{m-1}} d\eta.$$

Thus, we have constructed a solution of the Cauchy problem for the porous media equation with initial data of measure concentrated inside the $(n-1)$-dimensional ellipsoid $E_n(0)$. The initial condition (3.9) is fulfilled in the sense of distributions. The solution of the Cauchy problem (1.1) and (3.9) is unique within the appropriately defined class of its generalized solutions. This result follows from the general theorem of uniqueness [7].

The solution (3.5) is a far-reaching generalization of the source-type solution for the porous media equation. On the other hand, this solution generalizes the solution of the equation $u_t = \Delta(u^2)$ obtained in [10], which is quadratic in the space variables.

Now let $\gamma_{n-1} = \ldots = \gamma_{n-l} = 0$, but $\gamma_{n-l-1} > 0$, where $1 \leq l \leq n-2$. Then the initial distribution (1.2) corresponding to the solution (3.5) is a measure of the type

$$u_0 = K_{m,l}\Big(\prod_{j=1}^{n-l-1}\gamma_j\Big)^{-1/2}\Big[\Big(C-\sum_{j=1}^{n-l-1}\frac{x_j^2}{\gamma_j}\Big)_+\Big]^{\frac{2+(l+1)(m-1)}{2(m-1)}}\delta(x_{n-l})\dots\delta(x_n),$$

where

$$K_{m,l} = \Omega_{l+1}\int_0^1(1-\eta^2)^{\frac{1}{m-1}}\eta^l d\eta$$

(Ω_n is the surface area of the unit sphere in R^n). The singular support of this measure is $(n-l-1)$-dimensional (we agree that $n \geq 2$). In this case the solution (3.5) itself is invariant under the rotation group in the space R^{l+1} and therefore contains only $n-1$ free parameters.

Finally, if all $\gamma_1, \dots, \gamma_{n-1}$ are equal to zero, the solution (3.5) becomes invariant with respect to the complete rotation group in R^n and transforms into the well-known solution of G. I. Barenblatt [5] corresponding to the initial data

$$u_0 = K_{m,n-1}C^{\frac{2+n(m-1)}{2(m-1)}}\delta(x).$$

Here $\delta(x)$ is the delta function in R^n. The solution incidentally acquires self-similarity, which is not characteristic of the solutions (3.5) if at least one of the constants γ_j is nonzero.

4. Let us turn to the fast diffusion equation, i.e., to Eq. (1.1), in which $0 < m < 1$. Operating by analogy with Section 3, we will obtain a family of solutions (1.1) of the form

$$u = \Big(\prod_{i=1}^{n}\alpha_i\Big)^{-1}\Big(C+\sum_{i=1}^{n}\frac{x_i^2}{\alpha_i^2}\Big)^{\frac{-1}{1-m}}. \tag{4.1}$$

Here the functions $\alpha_i(t)$ are again related by Eqs. (3.6), however, the dependence $\alpha_n(t)$ will differ for the cases $m < 1-2/n$ and $m \geq 1-2/n$. We first consider the case

$$m \geq 1 - 2/n \tag{4.2}$$

(if $n = 2$, inequality (4.2) must be strict). Then the function $\alpha_n(t)$ satisfying the condition $\alpha_n(0) = 0$ is given by the equality

$$\int_0^{\alpha_n}\beta^m\Big[\prod_{j=1}^{n-1}(\beta^2+\gamma_j)\Big]^{-\frac{1-m}{2}}d\beta = \frac{2mt}{1-m}. \tag{4.3}$$

In view of (4.2), the function $\alpha_n(t)$ is defined and positive for all $t \to 0$, while $\alpha_n \to \infty$ as $t \to \infty$.

If all inequalities (3.7) are strict and $\gamma_{n-1} > 0$, then the relations (3.6), (4.1), and (4.3) determine an n-parameter family of solutions of the fast diffusion equation. The initial distribution in these solutions is the measure

$$u_0 = L_{m,0}\Big(\prod_{j=1}^{n-1}\gamma_j\Big)^{-1/2}\Big(C+\sum_{j=1}^{n-1}\frac{x_j^2}{\gamma_j}\Big)^{-\frac{1+m}{2(1-m)}}\delta(x_n), \tag{4.4}$$

where

$$L_{m,0} = 2\int_0^\infty(1+\eta^2)^{-\frac{1}{1-m}}.$$

As opposed to the solutions considered in Section 3, the support of measure (4.4) is not compact. However, if inequality (4.2) is strict, then the density of this measure concentrated at the hyperplane $x_n = 0$ is a function of the class $L_1(R^{n-1})$. In this case the solution (4.1) has a finite $L_1(R^n)$-norm for all $t > 0$, and the law of mass conservation (2.2) is valid for it. If $m = 1-2/n$ and $n \geq 3$, then the "mass" of the solution (4.1) is infinite.

The solution (4.1) of the fast diffusion equation with initial data of the type (4.4) corresponds to the case of "common position" in space of the parameters $\gamma_1, \ldots, \gamma_{n-1}$. The other extreme case, when $\gamma_1 = \ldots = \gamma_{n-1} = 0$, corresponds to the source-type solution (4.1). Here initial data are as follows

$$u_0 = L_{m,n-1} C^{\frac{2-n(1-m)}{2(1-m)}} \delta(x),$$

where

$$L_{m,n-1} = \Omega_n \int_0^\infty (1+\eta^2)^{-\frac{1}{1-m}} \eta^{n-1} d\eta.$$

In this case it should be assumed that inequality (4.2) is strict. In the extreme case (m = 1-2/n) the fast diffusion equation has no solutions of the source type.

If n = 2, then the general situation is represented only by the two above cases. For $n \geq 3$ intermediate cases can occur when $\gamma_{n-1} = \ldots = \gamma_{n-1} = 0$, but $\gamma_{n+l-1} > 0$ for $l \in [1, n-2]$. Here the solution (4.1) acquires the property of symmetry with respect to rotations in the space R^{l+1}, but simultaneously it loses l of the n free parameters.

Let now the exponent m satisfy the following inequalities instead of (4.2)

$$0 < m < 1 - 2/n$$

(presumably, it is possible only when $n \geq 3$). In this case Eq. (1.1) again possesses exact solutions of the form (4.1); however, the dependence of α_n on t will become other than (4.3)

$$\int_{\alpha_n}^\infty \beta^m \Big[\prod_{j=1}^{n-1} (\beta^2 + \gamma_j)\Big]^{-\frac{1-m}{2}} d\beta = \frac{2m}{1-m}(\tau - t). \tag{4.5}$$

where

$$\tau = \tau(m, \gamma_1, \ldots, \gamma_{n-1}) = \frac{1-m}{2m} \int_0^\infty \beta^m \Big[\prod_{j=1}^{n-1} (\beta^2 + \gamma_j)\Big]^{-\frac{1-m}{2}} d\beta;$$

we assume for definiteness that $\gamma_1 \geq \ldots > \gamma_{n-1} > 0$. Here, as before, $\alpha_n(0) = 0$; at the same time $\alpha_j(0) = \gamma_j^{1/2} > 0$ for $j = 1, \ldots, n-1$. Therefore, the initial distribution u_0 again has the form (4.4). However, unlike (4.2), now the solution (4.1) is not positive for all $t > 0$. From (3.6) and (4.5) it is obvious that as $t \to \tau$ the function u(x, t) vanishes uniformly in $x \in R^n$.

5. Consider again the solutions (3.5) of the porous media equation, but now we will not assume that all constants γ_j in (3.6) are nonnegative. The initial data in such solutions are a measure with noncompact support, whose boundary is the surface of the second order in R^{l+1} with nondegenerate quadratic form, where $l = 1, \ldots, n-2$ (we infer that $n \geq 2$). It can be easily proved that all such solutions fail over a finite time. Without analyzing the variety of possible consequences, we shall restrict our attention to examples of exact solutions of the Boussinesq equation

$$\frac{\partial u}{\partial t} = \Big(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\Big)(u^2). \tag{5.1}$$

Equation (5.1) is a simple multidimensional model of the porous media equation. The Boussinesq equation is of independent significance because it describes approximately the process of planned filtration of an incompressible liquid in homogeneous soil above a horizontal water table. The nonnegative function u(x, y, t) determines the level of ground water.

One of the solutions of Eq. (5.1) is of the form

$$u = \frac{3[x^2 \sin^2 \mu(t) - y^2 \cos^2 \mu(t)]_+}{32 \sin^3 2\mu(t)}, \tag{5.2}$$

where the function $\mu(t)$ is implicitly given by the equality

$$4\mu - \sin 4\mu = 3t.$$

This function is monotonically increasing over the interval $[0, 2\pi/3]$ from $\mu = 0$ to $\pi/2$. Here the Cauchy data are the measure

$$u_0 = \frac{1}{256}(x^3)_+\delta(y), \tag{5.3}$$

which is concentrated at the ray $x \geq 0$, $y = 0$. The free boundaries in the solution (5.2) are two half-lines $y = \pm x\mathrm{tg}\mu(t)$, $x \geq 0$. When $t \to 2\pi/3 - 0$, the function $u(x, y, t)$ becomes infinite simultaneously at all points of the half-plane $x > 0$.

It should be noted that the solution (5.2) is self-similar (invariant with respect to the dilatation transformation $x' = ax$, $y' = ay$, $u' = a^2u$, where $a = \mathrm{const} > 0$). This self-similarity is of another nature than in the source-type solution, since one of the invariants of the transformation is the time t.

Equation (5.1) admits another dilatation transformation $t' = bt$, $u' = b^{-1}u$, where $b = \mathrm{const} > 0$. Using this property, one can obtain from the solution (5.2) a one-parameter family of solutions with initial data of the type of (5.3), where the coefficient 1/256 should be replaced by 1/256b.

Another family of exact solutions of the Boussinesq equation corresponds to initial data of the form

$$u_0 = \left[\left(\frac{x^2}{\gamma} - C\right)_+\right]^{3/2}\delta(y)$$

(γ and C are arbitrary positive constants). The free boundaries in these solutions are hyperbolas at the plane x, y. Without writing down these solutions, we note only that for each of them there is such $t_* = t_*(\gamma, C)$ that $u \to \infty$ as $t \to t_*-0$ at all points of the half-plane $x > 0$ at once.

6. It appears that apart from solutions of the type of (5.2), the Boussinesq equation has solutions in which the function u is quadratic in y, but linear in x. One of the solutions is given by the formula

$$u = (3A^2t^{1/3} + At^{-1/3}x - y^2/12t)_+\,, \tag{6.1}$$

where $A = \mathrm{const} > 0$. Solution (6.1) corresponds to initial data of the type

$$u_0 = 8 \cdot 3^{-1/2}[(Ax)_+]^{3/2}\delta(y). \tag{6.2}$$

The free boundary in this solution is parabolic. The solution itself exists for all $t > 0$, although the initial measure (6.2) is infinite.

Comparing the solutions of the Cauchy problems (5.3) and (6.2) for Eq. (5.1), one can observe that as $t \to \infty$ the order of growth of the density of the initial measure concentrated at the half-axis $x > 0$ significantly influences the global solvability for t.

There are also analogs to the solution (6.1) for the general porous media equation (1.1), but we shall not discuss them.

7. In conclusion we consider several exact solutions of the equation

$$u_t = \Delta \ln u, \tag{7.1}$$

which is a limiting case of the fast diffusion equation. With $n = 3$ Eq. (7.1) describes the evolution of the density of an electron beam obeying a Maxwell distribution, while at $n = 2$ it describes the process of spreading of an ultrathin liquid film under the action of van der Waals forces. Moreover, Eq. (7.1) has applications in geometry.

Having confined our consideration to the cases $n = 2$ and 3, we introduce the notation $x_1 = x$, $x_2 = y$, $x_3 = z$. One of the solutions of Eq. (7.1) is the function

$$u = \frac{2\mathrm{sh}t\,\mathrm{ch}t}{x^2\mathrm{sh}^2t + y^2\mathrm{ch}^2t}. \tag{7.2}$$

The solution corresponds to the initial data

$$u_0 = \frac{2\pi\delta(x)}{|y|}. \tag{7.3}$$

The solution (7.2) exists for all $t < 0$, while as $t \to \infty$ it has the stationary limit $\hat{u} = 2(x^2 + y^2)^{-1}$. Furthermore, the estimate

$$\frac{u - \hat{u}}{\hat{u}} = O(e^{-t})$$

holds uniformly in $(x, y) \in R^2$ as $t \to \infty$. The function $\hat{u}$ is a unique (to within a constant multiplier) stationary solution of Eq. (7.1) in the plane, which is invariant under rotations. It is interesting to note that the integrals over any ring $R_{a,b} = \{x, y: a^2 < x^2 + y^2 \le b^2\}$ of the initial function (7.3) and of the limiting stationary solution $\hat{u}$ coincide:

$$\iint_{R_{a,b}} u_0\, dx\, dy = \iint_{R_{a,b}} \hat{u}\, dx\, dy = 4\pi \ln(b/a).$$

Let us consider Eq. (7.1) in a three-dimensional space. The equation has the solution

$$u = \frac{2\sin t\cos t}{x^2 + \sin^2 t(y^2 + z^2)}. \tag{7.4}$$

The initial data are

$$u_0 = \frac{2\pi\delta(x)}{(y^2 + z^2)^{1/2}}. \tag{7.5}$$

The solution (7.4) is nonnegative in the layer $R^3 \times (0, \pi/2)$ and becomes zero at $t = \pi/2$. Along with (7.4) we shall consider a spherically symmetrical solution of Eq. (7.1) with the same "lifetime" $\pi/2$:

$$\bar{u} = \frac{\pi - 2t}{x^2 + y^2 + z^2}.$$

The estimate

$$\frac{u - \bar{u}}{\bar{u}} = O(\pi/2 - t)$$

holds uniformly in $(x, y, z) \in R^3$ as $t \to \pi/2\text{-}0$. This means that the solution (7.4) can become symmetrized by the time it vanishes. It is noteworthy that in this case the integrals of the initial functions (7.5) and $\bar{u}_0 = \bar{u}(x_1, y, z, 0)$ are found to coincide for each sphere $B_a = \{x, y, z : x^2 + y^2 + z^2 < a^2\}$

$$\iiint_{B_a} u_0\, dxdydz = \iiint_{B_a} \bar{u}_0\, dxdydz = 4\pi^2 a.$$

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REFERENCES

1. Ya. B. Zel'dovich and A. S. Kompaneets, "Theory of heat propagation with heat conductivity depending on temperature," in: Collected Papers of the 70th Anniversary of the Birth of Academician A. F. Ioffe [in Russian], Moscow, (1950), pp. 61-71.
2. A. S. Kalashnikov, "Some problems of qualitative theory of nonlinear degenerate second-order parabolic equations," in: Usp. Mat. Nauk, **42**, No. 2, 135-176 (1987).
3. E. S. Sabinina, "A class of quasilinear parabolic equations which cannot be solved for the time derivative," Sib. Mat. Zh., **6**, No. 5, 1074-1100 (1965).
4. A. A. Samarskii, V. A. Galaktionov, S. P. Kurdyumov, and A. P. Mikhailov, Peaking Regimes in Problems for Quasilinear Parabolic Equations [in Russian], Nauka, Moscow (1987).
5. G. I. Barenblatt, "On some unsteady motions of liquid and gas in a porous medium," Prikl. Mat. Mekh., **16**, No. 1, 67-78 (1952).
6. A. Friedman and S. Kamin, "The asymptotic behavior of gas in an n-dimensional porous medium," Trans. Am. Math. Soc., **262**, 551-563 (1980).
7. M. Pierre, "Uniqueness of the solutions of $u_t = \Delta\phi(n) = 0$ with initial datum of measure," in: Nonlinear Analysis (Theory, Methods and Applications), Vol. 5, No. 2 (1982), pp. 175-187.
8. A. M. Meirmanov and V. V. Pukhnachov, "Lagrangian coordinates in the Stefan problem," in: Continuum Dynamics [in Russian], Novosibirsk, Vol. 7 (1980), pp. 90-111.
9. M. Gurtin, R. S. MacCamy, and E. A. Socolovsky, "A coordinate transformation for the porous media equation that renders the free boundary stationary," Q. Appl. Math., **42**, 345-357 (1984).
10. S. S. Titov and V. A. Ustinov, "Investigation of polynomial solutions of the gas filtration equation with an integer adiabatic exponent of adiabate," in: Approximate Methods of Solving Free-Boundary Problems in Continuum Mechanics [in Russian], Sverdlovsk, (1985), 64-70.
11. J. R. King, "Exact multidimensional solutions to some nonlinear diffusion equations," Q. J. Mech. Appl. Math., **46**, Part 3, 419-436 (1993).